

Mode regularization, time slicing, Weyl ordering, and phase space path integrals for quantum-mechanical nonlinear sigma models

Fiorenzo Bastianelli*

Dipartimento di Fisica, Università di Modena, via Campi 213/A, I-41100, Modena, Italy

Koenraad Schalm[†] and Peter van Nieuwenhuizen[‡]

Institute for Theoretical Physics, State University of New York at Stony Brook, Stony Brook, New York 11794-3840

(Received 16 January 1998; published 30 June 1998)

A simple, often invoked, regularization scheme of quantum mechanical path integrals in curved space is mode regularization: one expands fields into a Fourier series, performs calculations with only the first M modes, and at the end takes the limit $M \rightarrow \infty$. This simple scheme does not manifestly preserve reparametrization invariance of the target manifold: particular noncovariant terms of order \hbar^2 must be added to the action in order to maintain general coordinate invariance. Regularization by time slicing requires a different set of terms of order \hbar^2 which can be derived from Weyl ordering of the Hamiltonian. With these counterterms both schemes give the same answers to all orders of loops. As a check we perform the three-loop calculation of the trace anomaly in four dimensions in both schemes. We also present a diagrammatic proof of Matthews' theorem that phase space and configuration space path integrals are equal. [S0556-2821(98)03614-5]

PACS number(s): 04.62.+v

I. INTRODUCTION

Quantum mechanical path integrals in curved target spaces have various applications. Two decades ago they were used to quantize collective coordinates of solitons [1], and one decade ago to calculate chiral [2] and trace [3,4] anomalies in quantum field theories. In the former case, the path integrals were regularized by expanding the fluctuations about the solitons into normal modes, and cutting the sum over modes off at a maximum energy or at a maximum number of modes [5]. In the trivial vacuum, on the other hand, two methods have been widely used: the time discretization method¹ which has been pioneered by Dirac and Feynman, and the mode truncation method already described in [6]. In the literature, both methods have been used on a par.

In this article we point out the relation and the differences between the two methods. Both the time discretization method and the mode truncation method are particular regularization schemes which define the path integral. One expects therefore that the results for physical objects, for example the transition element $T = \langle x_f^\mu, t_f | x_i^\mu, t_i \rangle$, can be obtained from path integrals whose actions differ at most by finite local counterterms. In quantum gauge field theories, one can fix such ambiguities by requiring that Ward identities be satisfied. Similarly, for theories in curved target space

which are general coordinate invariant one can require that the transition element be a ‘bi-scalar’ (a scalar in x_f^μ and x_i^μ), but this fixes the ambiguities only up to covariant terms, namely a term proportional to the scalar curvature R . Only an experiment, for example the measurement of the trace anomaly, can fix the coefficient of this term. One can also view the Hamiltonian operator \hat{H} as an observable, and by writing $T = \langle x_f^\mu, t_f | x_i^\mu, t_i \rangle = \langle x_f^\mu | \exp[(-\beta/\hbar)\hat{H}] | x_i^\mu \rangle$ where $\beta = t_f - t_i$ in Euclidean time, one can fix the ambiguities in T by requiring that it satisfy the Schrödinger equation with \hat{H} as the Hamiltonian. If \hat{H} itself contains no term proportional to R , then the action will contain a term $-\frac{1}{8}R$ as we will show, and T will have an R term in the exponent with coefficient $-\frac{1}{12}$. If \hat{H} contains a term αR , the coefficient of R in T will be $-(\frac{1}{12} + \alpha)$. In general different regularization schemes can lead to different finite local counterterms of any order in \hbar , but finite local counterterms of order \hbar and \hbar^2 only are generated in the path integral action if one considers different operator orderings in the Hamiltonian. A Hamiltonian which is a target space scalar² is

$$\hat{H} = \frac{1}{2} g^{-1/4}(\hat{x}) \hat{p}_\mu g^{1/2}(\hat{x}) g^{\mu\nu}(\hat{x}) \hat{p}_\nu g^{-1/4}(\hat{x}) \quad (1)$$

but changing the order of the operators will in general de-

*Also at INFN, Sezione di Bologna, Italy. Email address: bastianelli@imoax1.unimo.it

[†]Email address: konrad@insti.physics.sunysb.edu

[‡]Email address: vannieu@insti.physics.sunysb.edu

¹Time discretization in the sector with solitons is complicated because the canonical momenta (of the non-zero modes) satisfy the equal-time commutation relation $[\pi(x, t), \varphi(y, t)] = -i\hbar[\delta(x - y) - \phi'_{sol}(x)\phi'_{sol}(y)/M_{sol}]$ where $\phi_{sol}(x)$ is the classical soliton solution with mass M_{sol} .

²The generator for target space diffeomorphisms $\hat{x}^\mu \rightarrow \hat{x}^\mu + \xi^\mu(\hat{x})$ is a sum of an orbital part $\hat{G}^{orb}(\xi) = (1/2i\hbar)[\hat{p}_\mu \xi^\mu(\hat{x}) + \xi^\mu(\hat{x}) \hat{p}_\mu]$ and a spin part $\hat{G}^{spin}(\xi) = (2\partial_\mu \xi^\lambda g_{\lambda\nu} - \xi^\lambda \partial_\lambda g_{\mu\nu}) \partial/\partial g_{\mu\nu}$. Closure of the commutator algebra fixes $\hat{G}^{spin}(\xi)$. An example of a target space diffeomorphism scalar is Eq. (1), namely $[\hat{G}^{orb}(\xi) + \hat{G}^{spin}(\xi), \hat{H}] = 0$, as one may check by an explicit (tedious) calculation.

stroy the invariance under target space diffeomorphisms. If the ambiguities due to using different regularization schemes correspond to the ambiguities due to different operator orderings, the terms of order \hbar^3 and higher in the Hamiltonian should be the same in all cases. This conclusion is corroborated by the observation that in these non-linear sigma models, N -loop graphs are convergent by power counting for $N \geq 3$.

In the time discretization method, a very clear connection between \hat{H} and the transition element exists if one uses phase space path integrals (Feynman's approach) instead of configuration space path integrals (Dirac's approach), and rewrites the Hamiltonian in Weyl ordered form. Then one may replace the Weyl ordered operator $\hat{H}_W(\hat{p}, \hat{x})$ in the kernel by the corresponding function at the midpoint (Berezin's theorem):

$$\int dp \langle x_1 | \exp\left(-\frac{\epsilon}{\hbar} \hat{H}_W\right) | p \rangle \langle p | x_2 \rangle \rightarrow \exp\left[-\frac{\epsilon}{\hbar} H\left(p, \frac{x_1 + x_2}{2}\right)\right] \int dp \langle x_1 | p \rangle \langle p | x_2 \rangle. \quad (2)$$

The Weyl ordering of \hat{H} in Eq. (1) leads to the following local finite correction to the naive Hamiltonian $H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu$ to be used in the path integral:

$$\Delta V_W = \frac{\hbar^2}{8} (R + g^{\mu\nu} \Gamma_{\mu\rho}{}^\sigma \Gamma_{\nu\sigma}{}^\rho), \quad (3)$$

where $\Gamma_{\mu\nu}{}^\sigma$ are the Christoffel symbols.³ This counterterm has been extensively discussed and used in the literature [7].

Both in the time discretization and mode truncation methods, one writes the paths as $x^\mu(\tau) = x_{bg}^\mu(\tau) + q^\mu(\tau)$ for $-1 \leq \tau \leq 0$, where $x_{bg}^\mu(\tau)$ is a solution of the free field equations with the correct boundary conditions [$x_{bg}^\mu(\tau) = x_f^\mu + \tau(x_f^\mu - x_i^\mu)$] and one expands the quantum fluctuations $q^\mu(\tau)$ into a complete set of eigenfunctions of the free field equations with vanishing boundary values and integrates over a finite number of coefficients q_m^μ . The difference between the time discretization method and mode regularization is that in the latter one uses the continuum action and naive continuum Feynman rules for a finite number of modes and one chooses a measure for the integrals over these modes which is usually normalized such that it reproduces the standard result for the free particle if the interaction part of the action, S_{int} , is set to zero. In the time discretization method, on the other hand, one uses a discretized action whose discretized Feynman rules are derived (in explicit form [8]) from the Hamiltonian starting point.

Up to this point we have made fairly obvious statements. Both regularization schemes are well defined, and this is how

they are used in the literature. However, it is already clear that this cannot be the whole truth, because with time slicing one has the nontrivial ΔV_W in Eq. (3), whereas with mode truncation such a ΔV seems at first to be absent. The precise way in which also mode truncation leads to a ΔV will be presented in Sec. II. We will see that one needs the following counterterm for mode regularization:

$$\Delta V_{MR} = \frac{\hbar^2}{8} \left(R - \frac{1}{3} g^{\mu\nu} g^{\alpha\beta} g_{\gamma\delta} \Gamma_{\mu\alpha}{}^\gamma \Gamma_{\nu\beta}{}^\delta \right). \quad (4)$$

As a check we shall obtain the correct trace anomaly in four dimensions from a three-loop calculation. In Riemann normal coordinates, the Christoffel symbols $\Gamma_{\mu\nu}{}^\rho$ in ΔV vanish at the origin of the coordinate system, but at the three-loop level one finds a contribution from ΔV by expanding each Christoffel symbol into $q^\tau \partial_\tau \Gamma_{\mu\nu}{}^\rho$ and contracting the two quantum fields. Since $\partial_\tau \Gamma_{\mu\nu}{}^\rho$ contains a part proportional to the Riemann tensor, it is clear that ΔV yields a nonvanishing contribution. Hence dropping ΔV in Riemann normal coordinates is incorrect.

If one eliminates the momenta by integrating over them in the time discretized path integral, one obtains N factors $(\det g^{\mu\nu})^{-1/2}$ at the N midpoint coordinates $\frac{1}{2}(q_{k+1}^\mu + q_k^\mu)$. Exponentiating these in the familiar Faddeev-Popov way one obtains what we have called ‘‘Lee-Yang ghosts,’’ namely commuting a^μ and anticommuting b^μ and c^μ ghost fields [4]. Phase space path integrals are free from ambiguities because they are finite. Clearly, upon the transition from phase space to configuration space, ambiguities are created; technically this is due to the fact that the momenta are replaced by the \dot{q} 's and ghosts, each of which introduces divergences and hence ambiguities into the theory.

In the phase space approach, the vertices are different [for example $\frac{1}{2} p_\mu g^{\mu\nu}(x) p_\nu$ instead of $\frac{1}{2} \dot{x}^\mu g_{\mu\nu}(x) \dot{x}^\nu$], as well as the propagators ($\langle p_\mu p_\nu \rangle$ is not proportional to $g_{\mu\alpha} g_{\nu\beta} \langle \dot{q}^\alpha \dot{q}^\beta \rangle$), but the transition element should be the same (Matthews' theorem). In Sec. III we present a graphical proof.

In Sec. IV we draw conclusions and we show that the coefficient of the R term in ΔV is scheme independent and equals $\frac{1}{8}$.

II. MODE REGULARIZATION

We now describe how one can define mode regularized path integrals in curved space. Ideally, one would like to derive mode regularization from first principles, i.e., starting from the transition amplitude defined as a matrix element of the evolution operator acting in the Hilbert space of physical states: $\langle x_f^\mu | \exp[(-\beta/\hbar) \hat{H}] | x_i^\mu \rangle$. However, this derivation looks quite difficult, so that we prefer to take a more pragmatic course of action, and attempt a direct definition of the mode regularized path integral. This definition will be supplemented by certain consistency requirements which we specify later on.

The transition amplitude can formally be written as follows:

³Our conventions for the Riemann and Ricci tensor are $R_{\rho\mu\nu}{}^\sigma = \partial_\rho \Gamma_{\mu\nu}{}^\sigma + \Gamma_{\rho\tau}{}^\sigma \Gamma_{\mu\nu}{}^\tau - (\rho \leftrightarrow \mu)$ and $R_{\rho\nu} = R_{\rho\mu\nu}{}^\mu$. Hence, at the linearized level, $R_{\mu\nu} = \frac{1}{2}(\partial_\mu \partial_\nu h - \partial_\mu h_\nu - \partial_\nu h_\mu + \square h_{\mu\nu})$ where $h = \eta^{\mu\nu} h_{\mu\nu}$ and $h_\mu = \eta^{\nu\rho} \partial_\rho h_{\mu\nu}$.

$$\langle x_f^\mu, t_f | x_i^\mu, t_i \rangle = \int_{x(-1)=x_i}^{x(0)=x_f} \tilde{\mathcal{D}}x \exp \left[-\frac{1}{\beta \hbar} S \right],$$

$$S = \int_{-1}^0 d\tau \frac{1}{2} g_{\mu\nu}(x) (\dot{x}^\mu \dot{x}^\nu + a^\mu a^\nu + b^\mu c^\nu),$$

$$\tilde{\mathcal{D}}x = \prod_{-1 < \tau < 0} d^D x(\tau) d^D a(\tau) d^D b(\tau) d^D c(\tau). \quad (5)$$

We have shifted and rescaled the time parameter $t - t_f = \beta \tau$, and since all terms in the action only depend on $\beta \hbar$, we set $\hbar = 1$ from now on. Note that β then counts the number of loops. We will evaluate the path integral in a perturbative expansion in β and in the coordinate displacements ξ^μ about the final point: $\xi^\mu = x_i^\mu - x_f^\mu$. Formally the path integral is a scalar since the action is a scalar and the ghost fields make up a scalar measure on the space of paths.⁴ Mode regularization will destroy this formal covariance. However, we will see that covariance can be recovered by adding a suitable noncovariant counterterm ΔV to the action S .

We start parametrizing

$$x^\mu(\tau) = x_{bg}^\mu(\tau) + q^\mu(\tau), \quad (6)$$

where $x_{bg}^\mu(\tau)$ is a background trajectory and $q^\mu(\tau)$ the quantum fluctuations. The background trajectory is taken to satisfy the free field equations of motion and is a function linear in τ connecting x_i^μ to x_f^μ in the chosen coordinate system, thus enforcing the boundary conditions

$$x_{bg}^\mu(\tau) = x_f^\mu - \xi^\mu \tau, \quad \xi^\mu = x_i^\mu - x_f^\mu. \quad (7)$$

Then the quantum fields $q^\mu(\tau)$ should vanish at the time boundaries and can be expanded into sines. For the Lee-Yang ghosts we use the same Fourier expansion; this may be considered as part of our definition of mode regularization.⁵ Hence

$$q^\mu(\tau) = \sum_{m=1}^{\infty} q_m^\mu \sin(\pi m \tau), \quad a^\mu(\tau) = \sum_{m=1}^{\infty} a_m^\mu \sin(\pi m \tau),$$

$$b^\mu(\tau) = \sum_{m=1}^{\infty} b_m^\mu \sin(\pi m \tau), \quad c^\mu(\tau) = \sum_{m=1}^{\infty} c_m^\mu \sin(\pi m \tau). \quad (8)$$

At this point the formal measure $\tilde{\mathcal{D}}x$ can be defined in terms of integration over the Fourier coefficients

⁴The factors $\Pi[\det g(x_i)]^{1/2} dx_i^\mu$ in the discretized path integral are target space scalars and exponentiating them leads to ghosts.

⁵Another argument to justify that the ghosts should be expanded into sines is that the classical solutions of their field equations are $a^\mu = b^\mu = c^\mu = 0$, and that the quantum fluctuations do not modify the boundary conditions of the classical solutions. In [8] it was shown that the results for the transition amplitude do not change if one uses cosines for the ghosts.

$$\tilde{\mathcal{D}}x = \mathcal{D}q \mathcal{D}a \mathcal{D}b \mathcal{D}c = A \prod_{m=1}^{\infty} \prod_{\mu=1}^D dq_m^\mu da_m^\mu db_m^\mu dc_m^\mu, \quad (9)$$

which fixes the path integral for a free particle up to the constant A [see also Eqs. (36) and (22)]:

$$\int \tilde{\mathcal{D}}x \exp \left[-\frac{1}{\beta} Q \right] = A,$$

$$Q = \int_{-1}^0 d\tau \frac{1}{2} \delta_{\mu\nu} (\dot{q}^\mu \dot{q}^\nu + a^\mu a^\nu + b^\mu c^\nu). \quad (10)$$

The constant A will be fixed later on from a consistency requirement. Note that limiting the integration over the number of modes to a finite number M gives a natural regularization of the path integral. This regularization resolves the ambiguities that show up in the continuum limit.

Now we expand the action about the final point x_f^μ and obtain

$$S = S_2 + S_{int} = S_2 + S_3 + S_4 + \dots \quad (11)$$

where

$$S_2 = \int_{-1}^0 d\tau \frac{1}{2} g_{\mu\nu} (\xi^\mu \xi^\nu + \dot{q}^\mu \dot{q}^\nu + a^\mu a^\nu + b^\mu c^\nu),$$

$$S_3 = \int_{-1}^0 d\tau \frac{1}{2} \partial_\alpha g_{\mu\nu} (q^\alpha - \xi^\alpha \tau) (\xi^\mu \xi^\nu + \dot{q}^\mu \dot{q}^\nu + a^\mu a^\nu + b^\mu c^\nu - 2\dot{q}^\mu \xi^\nu),$$

$$S_4 = \int_{-1}^0 d\tau \frac{1}{4} \partial_\alpha \partial_\beta g_{\mu\nu} (q^\alpha q^\beta + \xi^\alpha \xi^\beta \tau^2 - 2q^\alpha \xi^\beta \tau) \times (\xi^\mu \xi^\nu + \dot{q}^\mu \dot{q}^\nu + a^\mu a^\nu + b^\mu c^\nu - 2\dot{q}^\mu \xi^\nu). \quad (12)$$

All geometrical quantities, like $g_{\mu\nu}$ or $\partial_\alpha g_{\mu\nu}$, are evaluated at the final point x_f^μ , but for notational simplicity we do not exhibit this dependence. S_2 is taken as the free part and defines the propagators while S_{int} gives the vertices as usual. Therefore, the quantum perturbative expansion reads

$$\langle x_f^\mu, t_f | x_i^\mu, t_i \rangle = \int \tilde{\mathcal{D}}x \exp \left[-\frac{1}{\beta} S \right] = A e^{(-1/2\beta) g_{\mu\nu} \xi^\mu \xi^\nu} \langle e^{(-1/\beta) S_{int}} \rangle$$

$$= A e^{(-1/2\beta) g_{\mu\nu} \xi^\mu \xi^\nu} \left(\left\langle 1 - \frac{1}{\beta} S_3 - \frac{1}{\beta} S_4 + \frac{1}{2\beta^2} S_3^2 \right\rangle \right) + O(\beta^3/2).$$

(13)

Aiming at a two-loop computation, we have kept only those terms contributing up to $O(\beta)$, taking into account that $\xi^\mu \sim O(\sqrt{\beta})$, as follows from the exponential appearing in the last line of Eq. (13). The propagators that follow from S_2 are given by

$$\begin{aligned}\langle q^\mu(\tau)q^\nu(\sigma) \rangle &= -\beta g^{\mu\nu}(x_f)\Delta(\tau, \sigma) \\ \langle a^\mu(\tau)a^\nu(\sigma) \rangle &= \beta g^{\mu\nu}(x_f)^{\bullet\bullet}\Delta(\tau, \sigma) \\ \langle b^\mu(\tau)c^\nu(\sigma) \rangle &= -2\beta g^{\mu\nu}(x_f)^{\bullet\bullet}\Delta(\tau, \sigma)\end{aligned}\quad (14)$$

where Δ is regulated by the mode cutoff introduced below Eq. (10),

$$\Delta(\tau, \sigma) = \sum_{m=1}^M \left[-\frac{2}{\pi^2 m^2} \sin(\pi m \tau) \sin(\pi m \sigma) \right], \quad (15)$$

and has as continuum value

$$\Delta(\tau, \sigma) = \tau(\sigma+1)\theta(\tau-\sigma) + \sigma(\tau+1)\theta(\sigma-\tau). \quad (16)$$

[A dot on the left (right) of $\Delta(\tau, \sigma)$ indicates differentiation with respect to τ (σ).] Using standard Wick contractions, we computed

$$\left\langle -\frac{1}{\beta} S_3 \right\rangle = -\frac{1}{\beta} \frac{1}{4} \partial_\alpha g_{\mu\nu} \xi^\alpha \xi^\mu \xi^\nu, \quad (17)$$

$$\left\langle -\frac{1}{\beta} S_4 \right\rangle = \partial_\alpha \partial_\beta g_{\mu\nu} \left[\frac{\beta}{24} (g^{\mu\nu} g^{\alpha\beta} - g^{\mu\alpha} g^{\nu\beta}) - \frac{1}{24} (g^{\mu\nu} \xi^\alpha \xi^\beta + g^{\alpha\beta} \xi^\mu \xi^\nu - 2g^{\mu\alpha} \xi^\nu \xi^\beta) - \frac{1}{\beta} \frac{1}{12} \xi^\mu \xi^\nu \xi^\alpha \xi^\beta \right], \quad (18)$$

$$\begin{aligned}\left\langle \frac{1}{2\beta^2} S_3^2 \right\rangle &= \partial_\alpha g_{\mu\nu} \partial_\beta g_{\lambda\rho} \left[\frac{\beta}{96} (g^{\alpha\beta} g^{\mu\nu} g^{\lambda\rho} - 4g^{\alpha\rho} g^{\mu\nu} g^{\beta\lambda} - 6g^{\alpha\beta} g^{\mu\lambda} g^{\nu\rho} + 4g^{\alpha\rho} g^{\beta\mu} g^{\nu\lambda} + 4g^{\alpha\mu} g^{\beta\lambda} g^{\nu\rho}) \right. \\ &\quad + \frac{1}{48} (g^{\mu\lambda} g^{\nu\rho} \xi^\alpha \xi^\beta + 2(g^{\alpha\beta} g^{\mu\lambda} - g^{\alpha\lambda} g^{\mu\beta}) \xi^\nu \xi^\rho + (2g^{\alpha\lambda} g^{\beta\rho} - g^{\alpha\beta} g^{\lambda\rho}) \xi^\mu \xi^\nu + (2g^{\mu\beta} g^{\lambda\rho} - 4g^{\mu\lambda} g^{\beta\rho}) \xi^\alpha \xi^\nu) \\ &\quad \left. + \frac{1}{\beta} \frac{1}{96} (g^{\alpha\beta} \xi^\mu \xi^\nu \xi^\lambda \xi^\rho - 4g^{\alpha\lambda} \xi^\mu \xi^\nu \xi^\beta \xi^\rho + 4g^{\mu\lambda} \xi^\alpha \xi^\nu \xi^\beta \xi^\rho) + \frac{1}{\beta^2} \frac{1}{32} \xi^\alpha \xi^\mu \xi^\nu \xi^\beta \xi^\lambda \xi^\rho \right].\end{aligned}\quad (19)$$

This completes the calculation of the transition amplitude in the two-loop approximation using the mode regularized path integral.

At this point we should make contact with other schemes and test the consistency of our rules. To do that, we use our transition amplitude to obtain the time evolution of an arbitrary wave function

$$\Psi(x_f, t_f) = \int d^D x_i \sqrt{g(x_i)} \langle x_f^\mu, t_f | x_i^\mu, t_i \rangle \Psi(x_i, t_i). \quad (20)$$

We need the factor $\sqrt{g(x_i)}$ because the transition element is formally a bi-scalar as we explained before, but then $\Psi(x_f, t_f)$ is a scalar and hence also $\Psi(x_i, t_i)$, which in turn implies that the measure must be a scalar as well.

Since the transition amplitude (13) is given in terms of an expansion around the final point (x_f, t_f) , we Taylor expand also the wave function $\Psi(x_i, t_i)$ and the measure $\sqrt{g(x_i)}$ in Eq. (20) about this point:

$$\begin{aligned}\Psi(x_i, t_i) &= \Psi(x_f, t_f) - \beta \partial_t \Psi(x_f, t_f) + \xi^\mu \partial_\mu \Psi(x_f, t_f) \\ &\quad + \frac{1}{2} \xi^\mu \xi^\nu \partial_\mu \partial_\nu \Psi(x_f, t_f) + O(\beta^{3/2})\end{aligned}$$

$$\begin{aligned}\sqrt{g(x_i)} &= \sqrt{g(x_f)} \left(1 + \xi^\mu \Gamma_{\mu\alpha}{}^\alpha + \frac{1}{2} \xi^\mu \xi^\nu (\partial_\mu \Gamma_{\nu\alpha}{}^\alpha \right. \\ &\quad \left. + \Gamma_{\mu\alpha}{}^\alpha \Gamma_{\nu\beta}{}^\beta) + O(\xi^3) \right).\end{aligned}\quad (21)$$

We then perform the integration over $d^D x^\mu = d^D \xi^\mu$ in Eq. (20) and match the various terms. The leading term fixes A ,

$$\Psi = A(2\pi\beta)^{D/2} \Psi \rightarrow A = (2\pi\beta)^{-D/2}, \quad (22)$$

and the terms of order β give

$$\begin{aligned}\beta \left[-\partial_t \Psi + \frac{1}{2} \nabla^2 \Psi + \frac{1}{8} R \Psi - \frac{1}{32} g^{\mu\nu} g^{\alpha\beta} g^{\gamma\delta} \partial_\mu g_{\alpha\gamma} \partial_\nu g_{\beta\delta} \Psi \right. \\ \left. + \frac{1}{48} g^{\mu\nu} g^{\alpha\beta} g^{\gamma\delta} \partial_\mu g_{\alpha\gamma} \partial_\beta g_{\nu\delta} \Psi \right] = 0.\end{aligned}\quad (23)$$

This last equation means that the wave function Ψ satisfies the following Schrödinger equation at the final point (x_f^μ, t_f) :

$$-\partial_t \Psi = (H_0 + \Delta V_{eff}) \Psi = -\frac{1}{2} \nabla^2 \Psi + \Delta V_{eff} \Psi, \quad (24)$$

where the effective potential is given by

$$\begin{aligned}
\Delta V_{eff} &= -\frac{1}{8}R + \frac{1}{32}g^{\mu\nu}g^{\alpha\beta}g^{\gamma\delta}\partial_\mu g_{\alpha\gamma}\partial_\nu g_{\beta\delta} \\
&\quad - \frac{1}{48}g^{\mu\nu}g^{\alpha\beta}g^{\gamma\delta}\partial_\mu g_{\alpha\gamma}\partial_\beta g_{\nu\delta} \\
&= -\frac{1}{8}R - \frac{1}{8}g^{\mu\nu}\Gamma_{\mu\alpha}^\beta\Gamma_{\nu\beta}^\alpha \\
&\quad + \frac{1}{24}g^{\mu\nu}g^{\alpha\beta}g^{\gamma\delta}\partial_\mu g_{\alpha\gamma}\partial_\beta g_{\nu\delta} \\
&= -\frac{1}{8}R + \frac{1}{24}g^{\mu\nu}g^{\alpha\beta}g_{\gamma\delta}\Gamma_{\mu\alpha}^\gamma\Gamma_{\nu\beta}^\delta. \quad (25)
\end{aligned}$$

Clearly, to obtain the “free” Hamiltonian H_0 from a path integral, one should subtract the potential ΔV_{eff} , and thus use in Eq. (5) the following classical action:

$$-\frac{1}{\beta}S = \int_{-1}^0 d\tau \left(-\frac{1}{2\beta}g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu + \beta\Delta V_{eff}(x) \right). \quad (26)$$

With this counterterm mode regularization gives the same results as the time discretization scheme described in the Introduction. Note that Eq. (25) is different from the counterterm for time discretization. The difference is the term with $\frac{1}{24}$ in the fourth line of Eq. (25).

Finally, as a non-trivial test, which was one of our motivations to carry out the present research, we have computed the trace anomaly for a real conformal scalar field in four dimensions using mode regularization and the newly found counterterm. Anomalies in Fujikawa’s approach are given by the regulated trace of the Jacobian J of the symmetry transformation:

$$An = \lim_{\beta \rightarrow 0} \text{Tr} (J e^{(-\beta/\hbar)\mathcal{R}}). \quad (27)$$

A theory of consistent regulators \mathcal{R} exists [9] and it gives us for conformal scalar fields the Hamiltonian (1) minus an improvement potential $\hbar^2[(D-2)/8(D-1)]R$. The Jacobian for trace anomalies is proportional to unity. In mode regularization Eq. (27) translates into the evaluation of the expectation value of the Jacobian with respect to the path integral based on the formal transition element (5) including the just calculated counterterms and the improvement term. Taking the trace of Eq. (5) means first setting $x_f = x_i$ which gives that $x_{bg} = x_f$ and then integrating over x_f with $d^D x_f \sqrt{g(x_f)}$. As the measure of Eq. (5) includes the constant $A = (2\pi\beta)^{-D/2}$, the β -independent term of the right-hand side of Eq. (27) is given by the $(D/2+1)$ -loop term in the path integral. Expanding the metric in Riemann normal coordinates we find the first nonvanishing contributions at the two-loop level: graphs with topology of the number 8 and the background values of the various potential terms. From Eq. (18) with $\xi^\mu = 0$ one obtains⁶

$$\begin{aligned}
\text{Figure 8} + \bullet &= (-\beta\hbar)\frac{1}{6}\left(-\frac{1}{4}\right)R + (-\beta\hbar)\frac{(1-(D-2)/(D-1))}{8}R.
\end{aligned} \quad (28)$$

The result is the same as obtained from time slicing and yields the correct trace anomaly in $D=2$ dimensions.

At the three-loop level we find three regular graphs

$$\begin{aligned}
\text{Figure 8} &= \frac{1}{72}(-\beta\hbar)^2\left(-\frac{1}{6}R_{\mu\nu}^2\right),
\end{aligned} \quad (29)$$

$$\begin{aligned}
\text{Figure 9} &= \frac{1}{72}(-\beta\hbar)^2\left(-\frac{1}{6}R_{\mu\nu\lambda\rho}^2\right),
\end{aligned} \quad (30)$$

$$\begin{aligned}
\text{Figure 10} &= (-\beta\hbar)^2\left(\frac{1}{480}\nabla^2 R + \frac{1}{720}R_{\mu\nu\lambda\rho}^2 + \frac{1}{1080}R_{\mu\nu}^2\right),
\end{aligned} \quad (31)$$

as well as a graph coming from the potential terms:

$$\begin{aligned}
\text{Figure 11} &= -\frac{1}{6}(\beta\hbar)^2\left(\frac{1-(D-2)/(D-1)}{16}\nabla^2 R - \frac{1}{72}R_{\mu\nu\lambda\rho}^2\right).
\end{aligned} \quad (32)$$

⁶In Eqs. (28), (29), (30), (31) only the topology of the graphs is indicated; e.g., the figure eight in Eq. (28) stands for all possible graphs of that shape, including ghost loops.

There are only two differences with respect to time slicing: with time slicing, the factor of $-\frac{1}{6}$ in Eq. (30) becomes $-\frac{1}{4}$, and the factor of $-\frac{1}{12}$ in Eq. (32) becomes $-\frac{1}{48}$. These two modifications lead to the same final expression. Adding all contributions of connected and disconnected graphs we find the correct result

$An(\text{Weyl})$ (spin 0, $D=4$)

$$= \int \frac{d^4x}{(2\pi)^2} \sqrt{g(x)} \sigma(x) \frac{1}{720} (R_{\mu\nu\lambda\rho}^2 - R_{\mu\nu}^2 - \nabla^2 R), \quad (33)$$

where $\sigma(x)$ is the arbitrary function appearing in the Jacobian: $J = \sigma(x) \delta^D(x-y)$. This shows that mode regularization works after all.⁷ The fact that the right answer is obtained suggests that no new counterterms are needed in this scheme beyond those in Eq. (4). A complete three-loop calculation in arbitrary coordinates and with a non-vanishing ξ^μ could be used to test Eq. (20) at order β^2 . As we noted in the Introduction, however, all three- and higher-loop graphs are power-counting convergent. This means that at these orders in \hbar any consistent regularization scheme will yield the same answer. We therefore claim that with the presently found counterterm, the mode regularized path integral is consistent to all orders. This suggests that the difference is indeed due to operator-orderings. It would be interesting to justify mode regularization from first principles.

Before closing this section, we comment on how loop integrations in Feynman diagrams are done. The mode cutoff allows one to disentangle ambiguities that appear in the continuum limit of certain integrals over the Δ 's. Resorting to the mode regulated expression for Δ , one can use partial integration and take boundary terms into account if they are non-vanishing. Using partial integration repeatedly, one gets expressions containing only Δ 's, Δ^* 's and Δ^* 's which are unambiguous in the continuum limit, and computes them there. Useful identities obtained at the regulated level from Eq. (15) are

$$\begin{aligned} \Delta^*(\tau, \tau) + \Delta^*(\tau, \tau) &= \partial_\tau(\Delta(\tau, \tau)), \\ \Delta^*(\tau, \sigma) &= \Delta^*(\tau, \sigma). \end{aligned} \quad (34)$$

The first identity was used to compute Eq. (28). Here is a list of integrals that are easily computed using partial integration⁸ and whose values are *different* from the time discretization method:

⁷In [4] the noncovariant part of the counterterm (25) was missed. In Riemann normal coordinates this term affects only the coefficient of the $R_{\mu\nu\lambda\rho}^2$ term in the four dimensional trace anomaly, which was erroneously calculated.

⁸To obtain I_2 , use $(\Delta^*)(\Delta^*) = \frac{1}{2} \partial_\tau(\Delta^*)^2$, partially integrate, use $\Delta^* = \Delta^*$, and then use $(\Delta^*)(\Delta^*)^2 = \frac{1}{3} \partial_\sigma(\Delta^*)^3$. To obtain I_3 , write the integrand as $\Delta(\Delta^*) \frac{1}{2} \partial_\tau(\Delta^*)^2$, partially integrate, and use then $\Delta(\Delta^*)(\Delta^*)^2 = \Delta(\Delta^*)(\Delta^*)^2 = \Delta \frac{1}{3} \partial_\tau(\Delta^*)^3$. Then partially integrate once more.

$$I_1 = \int d\tau \tau (\Delta^* + \Delta^*)|_{\sigma=\tau} = 0,$$

$$I_2 = \int \int d\tau d\sigma (\Delta^*)(\Delta^*)(\Delta^*) = -\frac{1}{12},$$

$$I_3 = \int \int d\tau d\sigma \Delta(\Delta^*)(\Delta^*)(\Delta^*) = \frac{1}{180}. \quad (35)$$

Time discretization would give $I_1 = -\frac{1}{2}$, $I_2 = -\frac{1}{6}$ and $I_3 = \frac{7}{360}$.

The non-vanishing value of I_1 in time discretization is needed to compensate the explicit factors of $g^{-1/4}(x_i)$ appearing in the measure [see Eq. (36)], and together with I_2 it is responsible for the different counterterms required in the two regularization schemes. Finally, I_3 leads to the different values for the coefficient of $R_{\mu\nu\lambda\rho}^2$ in Eq. (30) in the two schemes. The counterterms with two Christoffel symbols are also different in both schemes but the final result for the transition element (and hence the trace anomaly) is the same. This confirms our approach to mode regularization.

III. PHASE SPACE PATH INTEGRALS IN CURVED SPACE AND MATTHEWS' THEOREM

In the phase space approach to path integrals in curved space the vertices and propagators are different from those used in the configuration space approach. For example, the leading terms in the action are $\frac{1}{2} p_\mu g^{\mu\nu}(x) p_\nu$ and $\frac{1}{2} \dot{x}^\mu g_{\mu\nu}(x) \dot{x}^\nu$, respectively, and the $\langle \dot{q}^\mu(\tau) \dot{q}^\nu(\sigma) \rangle$ propagator contains a Dirac delta function but the $\langle p_\mu(\tau) p_\nu(\sigma) \rangle$ propagator is completely regular. Although the vertices and propagators differ, the result for the transition element should be the same (Matthews' theorem¹⁰) [10], and the way this comes about is due to a new kind of ghosts [3,4]. The difference proportional to $\delta(\tau - \sigma)$ between the pp and $\dot{q}\dot{q}$ propagators turns out to be the propagator of those ghosts which we have called Lee-Yang ghosts after Lee and Yang, who were the first to point out that in the deformed harmonic oscillator (using arbitrary coordinates in one dimension) one needs extra $\delta(0)$ terms in the action or Hamiltonian [11]. The modern way to deal with these ill-defined objects $\delta(0)$ is to introduce new ghosts, just as in the familiar case of Faddeev-Popov ghosts for gauge theories. From the formal definition of the path integral, it is clear that Matthews' theorem should hold [12], but as always it is desirable

⁹In the time discretization scheme $\delta(\sigma - \tau)$ is a Kronecker delta function and $\theta(0) = \frac{1}{2}$. This leads to a fully consistent scheme as shown in [8]. For example, $\int \int d\sigma d\tau \delta(\sigma - \tau) \theta(\sigma - \tau) \theta(\sigma - \tau) = \int \int d\sigma d\tau \delta(\sigma - \tau) \theta(\sigma - \tau) \theta(\tau - \sigma) = \frac{1}{4}$ whereas mode regularization would give $\frac{1}{3}$ and $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ respectively. Using $\Delta(\tau, \sigma) = \sigma + \theta(\tau - \sigma)$, $\Delta^*(\tau, \sigma) = 1 - \delta(\tau - \sigma)$ and $\Delta^* = \delta(\tau - \sigma)$, the results below Eq. (35) follow.

¹⁰Matthews' theorem holds for connected graphs and originally only applied to nonderivative interactions.

to check this by a nontrivial calculation. Moreover, such a calculation in turn may reveal a relation between the Feynman graphs in both approaches which can lead to a diagrammatic proof of Matthews' theorem. With this in mind we calculate the two-loop correction to the transition element $\langle x_f^\mu, 0 | x_i^\mu, -\beta \rangle$, both using the phase space approach and the configuration space approach to path integrals.

Using time discretization (see footnote 9), the transition element is given in both phase- and configuration-space by

$$\begin{aligned} \langle x_f | \exp\left(-\frac{\beta}{\hbar} \hat{H}\right) | x_i \rangle \\ = \left[\frac{g(x_f)}{g(x_i)} \right]^{1/4} \frac{(-1/2\beta) g_{\mu\nu}(x_f) \xi^\mu \xi^\nu}{(2\pi\beta\hbar)^{D/2}} \left\langle \exp\left(-\frac{1}{\hbar} S_{int}\right) \right\rangle, \end{aligned} \quad (36)$$

where $(2\pi\beta\hbar)^{-D/2}$ is the usual Feynman measure and $\langle x_f |$ and $| x_i \rangle$ are both normalized as $\int d^D x \sqrt{g(x)} | x \rangle \langle x | = 1$. The factor $[g(x_f)/g(x_i)]^{1/4}$ in the measure is a direct result from this normalization.¹¹ Furthermore, $\tau = t/\beta$, and $x^\mu(\tau) = x_{bg}^\mu(\tau) + q^\mu(\tau)$, where $x_{bg}^\mu(\tau) = x_f^\mu + \tau(x_f^\mu - x_i^\mu)$ is the background solution of the free equations which satisfies the correct boundary conditions, and $q^\mu(\tau)$ are the quantum fluctuations.

The phase space interactions and propagators are given by

$$\begin{aligned} -\frac{1}{\hbar} S_{int}^{phase} = -\frac{1}{\beta\hbar} \int_{-1}^0 d\tau \frac{1}{2} [g^{\mu\nu}(x) - g^{\mu\nu}(x_f)] \tilde{p}_\mu \tilde{p}_\nu \\ + \frac{i}{\beta\hbar} \int_{-1}^0 d\tau \tilde{p}_\mu (x_f^\mu - x_i^\mu) - \frac{\beta}{\hbar} \int_{-1}^0 d\tau \Delta V_W, \end{aligned} \quad (37)$$

$$\begin{aligned} \langle \tilde{p}_\mu(\tau) \tilde{p}_\nu(\sigma) \rangle &= \beta\hbar g_{\mu\nu}(x_f) \sim -1, \\ \langle q^\mu(\tau) \tilde{p}_\nu(\sigma) \rangle &= -i\beta\hbar \delta_\nu^\mu [\tau + \theta(\sigma - \tau)] \sim i\Delta^*, \\ \langle q^\mu(\tau) q^\nu(\sigma) \rangle &= -\beta\hbar g^{\mu\nu}(x_f) \Delta(\tau, \sigma) \sim \Delta, \end{aligned} \quad (38)$$

where we recall

$$\Delta(\tau, \sigma) = \tau(\sigma + 1)\theta(\tau - \sigma) + \sigma(\tau + 1)\theta(\sigma - \tau). \quad (39)$$

We have rescaled $p_\mu(\sigma)$ into $(1/\beta)\tilde{p}_\mu(\sigma)$ in order that all propagators have a factor $\beta\hbar$ and all vertices a factor $(\beta\hbar)^{-1}$. The β and \hbar appear always together and count the

number of loops. The configuration space interactions and propagators are as in Sec. II, though regulated as in footnote 9:

$$\begin{aligned} -\frac{1}{\hbar} S_{int}^{conf} = -\frac{1}{\beta\hbar} \int_{-1}^0 d\tau \frac{1}{2} [g_{\mu\nu}(x) - g_{\mu\nu}(x_f)] (\dot{x}^\mu \dot{x}^\nu + a^\mu a^\nu \\ + b^\mu c^\nu) - \frac{\beta}{\hbar} \int_{-1}^0 d\tau \Delta V_W, \\ \langle q^\mu(\tau) q^\nu(\sigma) \rangle = -\beta g^{\mu\nu}(x_f) \Delta(\tau, \sigma) \sim \Delta \\ \langle q^\mu(\tau) \dot{q}^\nu(\sigma) \rangle = -\beta g^{\mu\nu}(x_f) [\tau + \theta(\sigma - \tau)] \sim \Delta^* \\ \langle \dot{q}^\mu(\tau) \dot{q}^\nu(\sigma) \rangle = -\beta g^{\mu\nu}(x_f) [1 - \delta(\tau - \sigma)] \sim \Delta^* \\ \langle a^\mu(\tau) a^\nu(\sigma) \rangle = \beta g^{\mu\nu}(x_f) \delta(\tau - \sigma) \sim -\Delta \\ \langle b^\mu(\tau) c^\nu(\sigma) \rangle = -2\beta g^{\mu\nu}(x_f) \delta(\tau - \sigma) \sim 2\Delta. \end{aligned} \quad (40)$$

From $\Delta^* = 1 - \delta(\tau - \sigma)$ and $\Delta = \delta(\tau - \sigma)$ it is clear that the sum of the $\langle \dot{q}\dot{q} \rangle$ and the ghost propagators is equal to minus the $\langle pp \rangle$ propagator:

$$\text{---}\cdot\text{---}\cdot + \text{---}\cdot\text{---}\cdot = -(\text{---}\cdot\text{---}\cdot). \quad (41)$$

Further, the $\langle qp \rangle$ and the $\langle q\dot{q} \rangle$ propagators are equal up to a factor i (note that $p = i\dot{x}$ is a field equation):

$$\text{---}\cdot\text{---}\cdot \times i = \text{---}\cdot\text{---}\cdot. \quad (42)$$

We are now ready to evaluate two-loop diagrams. The result for the $\xi^\mu = x_i^\mu - x_f^\mu$ independent part is given in Fig. 1. In the phase space approach, one finds derivatives of the inverse metric $g^{\mu\nu}$ which we have converted to derivatives of the metric $g_{\mu\nu}$ for the purpose of comparison. The relations A, B, F, and G follow immediately from Eqs. (41) and (42). Considering the identities C, D and E one must note that the phase space expression comes from both one- and two-vertex graphs. The identities then follow from rewriting Eq. (41) as

$$\text{---}\cdot\text{---}\cdot = -\text{---}\cdot\text{---}\cdot - \text{---}\cdot\text{---}\cdot$$

and replacing a “ghost” propagator on the phase space side by its value $(-\beta\hbar)$ times a delta-function, which pinches the two vertices. It is clear that the two-loop corrections in configuration space agree with those in phase space. We have also checked the ξ^μ dependent terms in the transition element and found complete agreement with Eq. (13). As, once again, all configuration space graphs at the three- and higher-loop level are unambiguous, the two-loop calculation suffices to show the equivalence between the phase- and configuration-space path integrals.

¹¹Inserting N sets of p -eigenstates and $N-1$ sets of x -eigenstates gives a factor $[g(x_f)g(x_i)]^{-1/4}$, but the phase space path integration over the p 's and the q 's gives an extra factor $g(x_f)^{1/2}$ because there is one more p than q 's. This extra factor is taken at the point x_f because we expanded the metric around this point. In the configuration space path integral, one integrates over the p 's, exponentiates the factors $g(x)$ by introducing ghosts, and then produces factors of $g(x)$ by integrating over the ghosts, with the same final result.

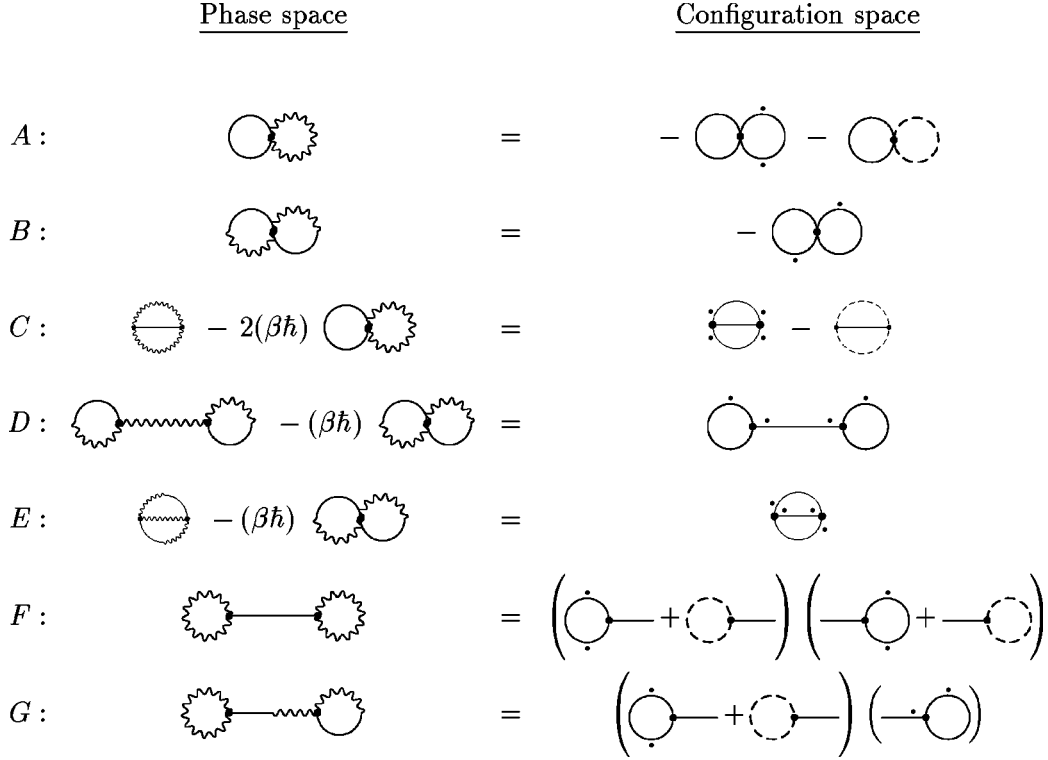


FIG. 1. Two-loop diagrammatic identities using time slicing. Each corresponds to the coefficient of a particular metric structure present in the two-loop partition function: A , $-\frac{1}{4} g^{\mu\nu} \square g_{\mu\nu}$; B , $-\frac{1}{4} g^{\alpha\mu} g^{\beta\nu} \partial_\alpha \partial_\beta g_{\mu\nu}$; C , $-\frac{1}{4} g^{\alpha\beta} g^{\mu\rho} g^{\nu\sigma} \partial_\alpha g_{\mu\nu} \partial_\beta g_{\rho\sigma}$; D , $-\frac{1}{2} g^{\alpha\mu} g^{\beta\rho} g^{\nu\sigma} \partial_\alpha g_{\mu\nu} \partial_\beta g_{\rho\sigma}$; E , $-\frac{1}{2} g^{\alpha\rho} g^{\beta\mu} g^{\nu\sigma} \partial_\alpha g_{\mu\nu} \partial_\beta g_{\rho\sigma}$; F , $-\frac{1}{8} g^{\alpha\beta} g^{\mu\nu} g^{\rho\sigma} \partial_\alpha g_{\mu\nu} \partial_\beta g_{\rho\sigma}$; and G , $-\frac{1}{2} g^{\alpha\mu} g^{\beta\nu} g^{\rho\sigma} \partial_\alpha g_{\mu\nu} \partial_\beta g_{\rho\sigma}$.

IV. CONCLUSIONS

The two regularization schemes considered in this article, mode regularization and time slicing, give the same answers at all orders of loops provided one adds specific order \hbar^2 counterterms to the action in the configuration space path integral. These terms are proportional to the curvature R but also to products of two Christoffel symbols and the latter are different for the two schemes. These extra terms of order \hbar^2 follow in the time slicing method from rewriting the Hamiltonian in Weyl-ordered form, but for mode regularization we could only determine them by requiring that the Schrödinger equation be satisfied. (This way of fixing the extra terms can also be applied to time slicing.) As a nontrivial check that these extra terms are indeed needed (and also a check on their explicit form) we evaluated the three-loop contributions to the trace anomaly in both schemes and indeed found the same (correct) answer.

We also considered phase space path integrals. Their loop graphs contain no divergences. We gave a diagrammatic proof of Matthews' theorem that phase and configuration space path integrals give the same transition element. A key observation was that the difference between the phase space propagator $\langle p(\sigma)p(\tau) \rangle$ and the configuration space propagator $\langle \dot{q}(\sigma)\dot{q}(\tau) \rangle$ is equal to the propagator for the Lee-Yang ghosts. The latter results when one exponentiates the factors $\sqrt{g(x)}$ which are produced by integrating out the momenta.

It remains to explain why the coefficient of the R term in ΔV is equal to $\frac{1}{8}$ in all "reasonable" regularization schemes. Let us call a scheme reasonable if it satisfies Matthews' theorem. Then the first two graphs in Fig. 1 should give the same result in configuration space as in phase space. The equality for graph B is always satisfied [see Eqs. (38) and (40)], but for graph A one finds the condition

$$\int_{-1}^0 d\tau \Delta(\tau, \tau) = \int_{-1}^0 d\tau \tau [\Delta(\tau, \tau) [\dot{\Delta}^*(\tau, \sigma) + \ddot{\Delta}(\tau, \sigma)]|_{\sigma=\tau}]. \quad (43)$$

Both time slicing and mode regularization satisfy this condition. In all such reasonable schemes the transition element with the naive action $S = \frac{1}{2} g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu$ will then contain a two-loop contribution $(\hbar^2/24)R$ just as the phase space calculation would give. The complete phase space Hamiltonian (37) contains, however, in addition the counterterm (3). The total R term in the Hamiltonian element is then $(-\hbar^2/12)R$ for any reasonable regularization scheme and the R term in the action to be used in the configuration space path integral has coefficient $\frac{1}{8}$.

ACKNOWLEDGMENT

The research of P.v.N. was supported by National Science Foundation Grant Phy 9722101.

- [1] For a review, see R. Rajaraman, *Solitons and Instantons* (North-Holland, Amsterdam 1982).
- [2] L. Alvarez-Gaumé and E. Witten, Nucl. Phys. **B234**, 269 (1984); L. Alvarez-Gaumé, in *Supersymmetry*, edited by K. Dietz *et al.* (Plenum Press, New York, 1984).
- [3] F. Bastianelli, Nucl. Phys. **B376**, 113 (1992).
- [4] F. Bastianelli and P. van Nieuwenhuizen, Nucl. Phys. **B389**, 53 (1993).
- [5] At the one-loop level mode regularization reproduces these results, but the energy cutoff does not; see A. Rebhan and P. van Nieuwenhuizen, Nucl. Phys. **B508**, 449 (1997). This can be explained by a new quantum principle for field theories with topological vacua; see H. Nastase, A. Rebhan, M. Stepha-nov, and P. van Nieuwenhuizen, hep-th/9802074.
- [6] R.P. Feynman and A.R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).
- [7] E. Tomboulis, Phys. Rev. D **12**, 1678 (1975); J.L. Gervais and A. Jevicki, Nucl. Phys. **B110**, 93 (1976). The latter authors show that Weyl ordering of the result of the first author leads to the ΔV_W in Eq. (3). Yang-Mills theory in the Coulomb gauge also contains a ΔV ; see J. Schwinger, Phys. Rev. **127**, 324 (1962); **130**, 406 (1963); N.H. Christ and T.D. Lee, Phys. Rev. D **22**, 939 (1980).
- [8] J. de Boer, B. Peeters, K. Skenderis, and P. van Nieuwenhuizen, Nucl. Phys. **B446**, 24 (1995); **B459**, 631 (1996).
- [9] A. Diaz, W. Troost, A. van Proeyen, and P. van Nieuwenhuizen, Int. J. Mod. Phys. A **4**, 3959 (1989); M. Hatsuda, W. Troost, A. van Proeyen, and P. van Nieuwenhuizen, Nucl. Phys. **B335**, 166 (1990); F. Bastianelli, A. van Proeyen, and P. van Nieuwenhuizen, Phys. Lett. B **253**, 67 (1991).
- [10] P.T. Matthews, Phys. Rev. **76**, 686 (1949); Y. Nambu, Prog. Theor. Phys. **7**, 131 (1952).
- [11] T.D. Lee and C.N. Yang, Phys. Rev. **128**, 885 (1962).
- [12] C. Bernard and A. Duncan, Phys. Rev. D **11**, 848 (1975). These authors verify Matthews' theorem by a path integral approach to second order in the coupling constant. However, they assume that all factors $\delta^4(0)$ which appear in their derivation may be set equal to zero. They also study as an example the Lagrangian $\mathcal{L} = -\frac{1}{2}(\partial_\mu\varphi)^2 - \frac{1}{2}m^2\varphi^2 - \frac{1}{4}\lambda(\partial_\mu\varphi)^4$. Again they can only prove Matthews' theorem if they set all $\delta^4(0)$ terms to zero. It would be satisfying if a path integral proof could be given without assuming that $\delta^4(0)$ is zero. Conceivably, careful discretization as well as the new ghosts of [3,4] are required.